# CS221 Problem Workout Solutions

Sept 16

## 1) Problem 1: Gradient and Gradient Descent

(i) Let $\phi(x) : \mathbb{R} \mapsto \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^d$. Consider the following objective function (a.k.a. loss function).

$$\text{Loss}(x, y, \mathbf{w}) = \begin{cases} 1 - 2(\mathbf{w} \cdot \phi(x))y & \text{if } (\mathbf{w} \cdot \phi(x))y \leq 0 \\ (1 - (\mathbf{w} \cdot \phi(x))y)^2 & \text{if } 0 < (\mathbf{w} \cdot \phi(x))y \leq 1 \\ 0 & \text{if } (\mathbf{w} \cdot \phi(x))y > 1, \end{cases}$$

where $y \in \mathbb{R}$. Compute the gradient $\nabla_{\mathbf{w}} \text{Loss}(x, y, \mathbf{w})$.

**Solution** We apply the rules to compute the gradient for each case separately, leading to the following piece-wise function for the gradient.

$$\nabla_{\mathbf{w}} \text{Loss}(x, y, \mathbf{w}) = \begin{cases} -2\phi(x)y & \text{if } (\mathbf{w} \cdot \phi(x))y \leq 0 \\ -2(1 - (\mathbf{w} \cdot \phi(x))y)\phi(x)y & \text{if } 0 < (\mathbf{w} \cdot \phi(x))y \leq 1 \\ 0 & \text{if } (\mathbf{w} \cdot \phi(x))y > 1 \end{cases} \tag{1}$$

(ii) Write out the Gradient Descent update rule for some function $\text{TrainLoss}(\mathbf{w}) : \mathbb{R}^d \mapsto \mathbb{R}$.

**Solution** $\mathbf{w} := \mathbf{w} - \eta \nabla_{\mathbf{w}} \text{TrainLoss}(\mathbf{w})$, where $\eta$ is the step size.

(iii) Let $d = 2$ and $\phi(x) = [1, x]$. Consider the following loss function.

$$\text{TrainLoss}(\mathbf{w}) = \frac{1}{2}\Big(\text{Loss}(x_1, y_1, \mathbf{w}) + \text{Loss}(x_2, y_2, \mathbf{w})\Big). \tag{2}$$

Compute $\nabla_w \text{TrainLoss}(\mathbf{w})$ for the following values of $x_1, y_1, x_2, y_2, \mathbf{w}$.

$$\mathbf{w} = \left[0, \frac{1}{2}\right],$$
$$x_1 = -2, \;\; y_1 = 1,$$
$$x_2 = -1, \;\; y_2 = -1.$$

**Solution**

$$\begin{aligned}\nabla_w \text{TrainLoss}(\mathbf{w}) &= \frac{1}{2}\nabla_{\mathbf{w}}\Big(\text{Loss}(x_1, y_1, \mathbf{w}) + \text{Loss}(x_2, y_2, \mathbf{w})\Big) \\ &= \frac{1}{2}\nabla_{\mathbf{w}}\text{Loss}(x_1, y_1, \mathbf{w}) + \frac{1}{2}\nabla_{\mathbf{w}}\text{Loss}(x_2, y_2, \mathbf{w})\end{aligned}$$

For each of the terms above, we plug in the expression for the gradient computed in part (i) above.

**Term one.** Note that $\phi(x_1) = [1, -2]$. Since $(\mathbf{w} \cdot \phi(x_1))y_1 = -1$, we consider the first piece (Case 1) in the gradient expression (Equation 1). We have

$$\begin{aligned}\nabla_{\mathbf{w}}\text{Loss}(x_1, y_1, \mathbf{w}) &= -2\phi(x_1)y_1 \\ &= [-2, 4]. \end{aligned} \tag{3}$$

**Term two.** Note that $\phi(x_2) = [1, -1]$. Similarly, $(\mathbf{w} \cdot \phi(x_2))y_2 = \frac{1}{2}$ taking us to Case 2 so

$$\begin{aligned}\nabla_{\mathbf{w}}\text{Loss}(x_2, y_2, \mathbf{w}) &= -2(1 - (\mathbf{w} \cdot \phi(x_2))y_2)\phi(x_2)y_2 \\ &= [1, -1]. \end{aligned} \tag{4}$$

Combining the terms,

$$\begin{aligned}\nabla_{\mathbf{w}}\text{TrainLoss}(\mathbf{w}) &= \frac{1}{2}\Big([-2, 4] + [1, -1]\Big) \\ &= \left[-\frac{1}{2}, \frac{3}{2}\right]. \end{aligned} \tag{5}$$

(iv) Perform two iterations of Gradient Descent to minimize the objective function $\text{TrainLoss}(\mathbf{w}) = \frac{1}{2}\Big(\text{Loss}(x_1, y_1, w) + \text{Loss}(x_2, y_2, w)\Big)$ with values for $x_1, y_1, x_2, y_2$ as above. Use initialization $\mathbf{w}^0 = \left[0, \frac{1}{2}\right]$ and step size $\eta = \frac{1}{2}$.

**Solution** Note that we have already computed $\nabla_{\mathbf{w}}\text{TrainLoss}(\mathbf{w})$ at the initialization point $\mathbf{w}^0$ in the question above.

$$\begin{aligned}\mathbf{w}^1 &= \mathbf{w}^0 - \eta\nabla_{\mathbf{w}}\text{TrainLoss}(\mathbf{w}) \text{ at } \mathbf{w}^0 \\ &= \left[0, \frac{1}{2}\right] - \left(\frac{1}{2}\right) \underbrace{\left(\frac{1}{2}\right)[-1, 3]}_{\text{From part (iii) above}} \\ &= \left[\frac{1}{4}, -\frac{1}{4}\right].\end{aligned}$$

Now we need to compute $\nabla_{\mathbf{w}}\text{Loss}(x_1, y_1, \mathbf{w})$ and $\nabla_{\mathbf{w}}\text{Loss}(x_2, y_2, \mathbf{w})$ at the new iterate $\mathbf{w}^1$.
We repeat the process we did for (iii) by applying the piece-wise defined gradient (Equation 1) to the two points, this time setting $\mathbf{w} = \mathbf{w}^1$.

**Term one.** Since $(\mathbf{w}^1 \cdot \phi(x_1))y_1 = \frac{3}{4}$, we have $\nabla_{\mathbf{w}}\text{Loss}(x_1, y_1, \mathbf{w}) = -2(1 - (\mathbf{w}^1 \cdot \phi(x_1))y_1)\phi(x_1)y_1 = [-\frac{1}{2}, 1]$. Note that we are now in Case 2 with respect to the piecewise definition of the gradient (Equation 1). When computing $\nabla_{\mathbf{w}}\text{Loss}(x_1, y_1, \mathbf{w})$ at $\mathbf{w}^0$, we were in Case 1.

**Term two.** $(\mathbf{w}^1 \cdot \phi(x_2))y_2 = -\frac{1}{2}$ taking us to Case 1, so $\nabla_{\mathbf{w}}\text{Loss}(x_2, y_2, \mathbf{w}) = -2\phi(x_2)y_2 = [2, -2]$.

Hence,

$$\begin{aligned}
\mathbf{w}^2 &= \mathbf{w}^1 - \eta\nabla_{\mathbf{w}}\text{TrainLoss}(\mathbf{w}) \text{ at } \mathbf{w}^1 \\
&= \left[\frac{1}{4}, -\frac{1}{4}\right] - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\left[-\frac{1}{2}, 1\right] + [2, -2]\right) \\
&= \left[-\frac{1}{8}, 0\right].
\end{aligned}$$

**2) Problem 2: Gradient computation**

(i) Let $\phi(x) : \mathbb{R} \mapsto \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^d$, and $f(x, \mathbf{w}) = \mathbf{w} \cdot \phi(x)$. Consider the following loss function.

$$\text{Loss}(x, y, \mathbf{w}) = \frac{1}{2}\max\{2 - (\mathbf{w} \cdot \phi(x))y, 0\}^2. \tag{6}$$

Compute its gradient $\nabla_{\mathbf{w}}\text{Loss}(x, y, \mathbf{w})$.

**Solution** Note that $\text{Loss}(x, y, \mathbf{w})$ can be written as the following piecewise defined function using the definition of max.

$$\text{Loss}(x, y, \mathbf{w}) = \begin{cases} \frac{1}{2}(2 - (\mathbf{w} \cdot \phi(x)y))^2 & \text{if } 2 - (\mathbf{w} \cdot \phi(x))y \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Using the chain rule, we get that the gradient is:

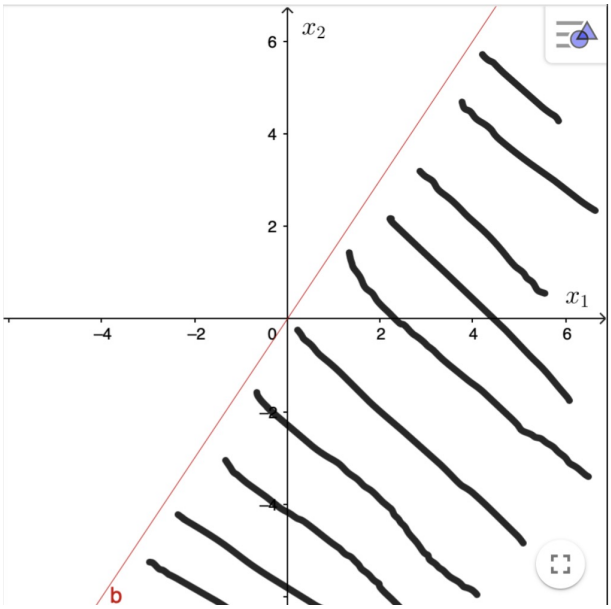
$$\nabla_{\mathbf{w}}\text{Loss}(x, y, \mathbf{w}) = \begin{cases} -(2 - \mathbf{w} \cdot \phi(x)y)\phi(x)y & \text{if } 2 - \mathbf{w} \cdot \phi(x)y \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

**3) Vector visualization**

Recall that we can visualize a vector $\mathbf{w} \in \mathbb{R}^d$ as a point in d-dimensional space. Let us now visualize some vectors in 2 dimensions on pen and paper.

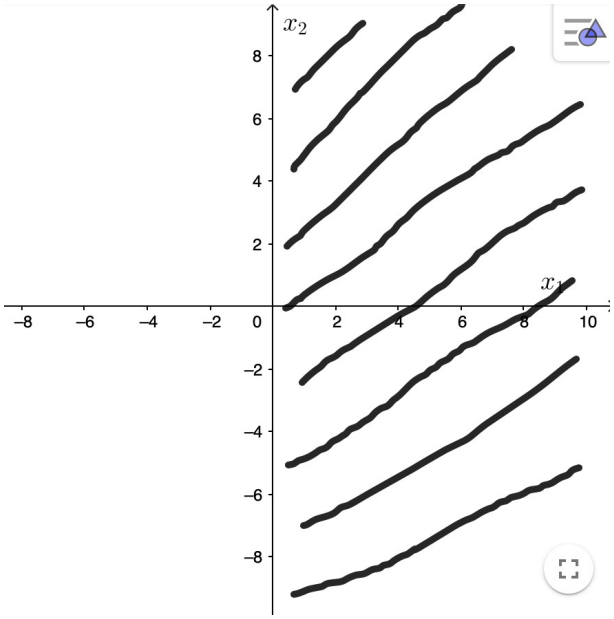
(i) Consider $\mathbf{x} \in \mathbb{R}^2$. Suppose we are interested only in vectors which have a positive dot product with $\mathbf{w} = [3, -2]$. Shade the part of the 2D plane that contains this set of vectors, i.e. $\mathbf{w} \cdot \mathbf{x} > 0$. Hint: It might help to write out the expression for the dot product and seeing the relation between $x_1$ and $x_2$ that leads to a positive dot product. You could also use the geometric interpretation of the dot product.

**Solution** $\mathbf{w} \cdot \mathbf{x} = 3x_1 - 2x_2 > 0$

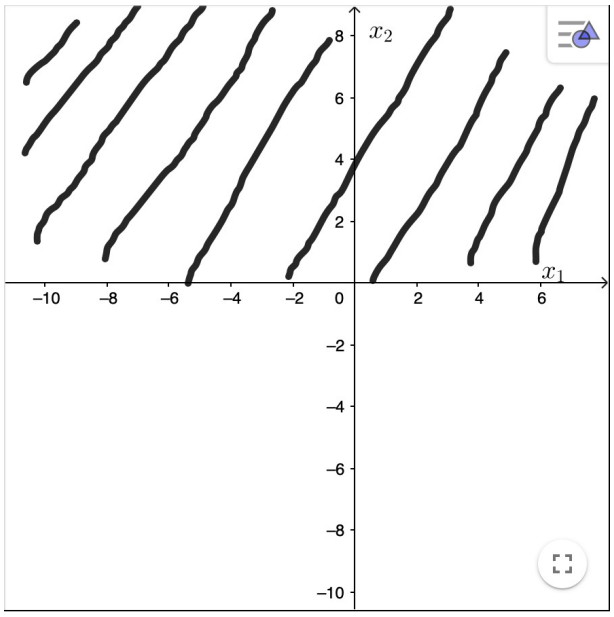


(ii) Repeat the above for $\mathbf{w} = [2, 0]$ and $\mathbf{w} = [0, 2]$.

**Solution** When $\mathbf{w} = [2, 0]$, $\mathbf{w} \cdot \mathbf{x} = 2x_1 > 0$

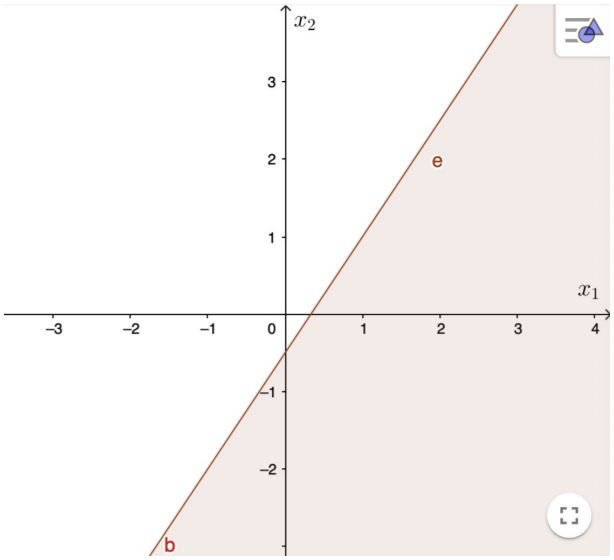



When $\mathbf{w} = [0, 2]$, $\mathbf{w} \cdot \mathbf{x} = 2x_2 > 0$



(iii) A small twist: visualize the set of vectors where $\mathbf{w} \cdot \mathbf{x} \geq 1$ for $\mathbf{w} = [3, -2]$.

**Solution** $\mathbf{w} \cdot \mathbf{x} = 3x_1 - 2x_2 \geq 1$, so $3x_1 - 2x_2 - 1 \geq 0$




Note that we get a line that is parallel to the one in (i) but shifted by a certain amount.

(iii) Consider the following element-wise inequality notation. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$,

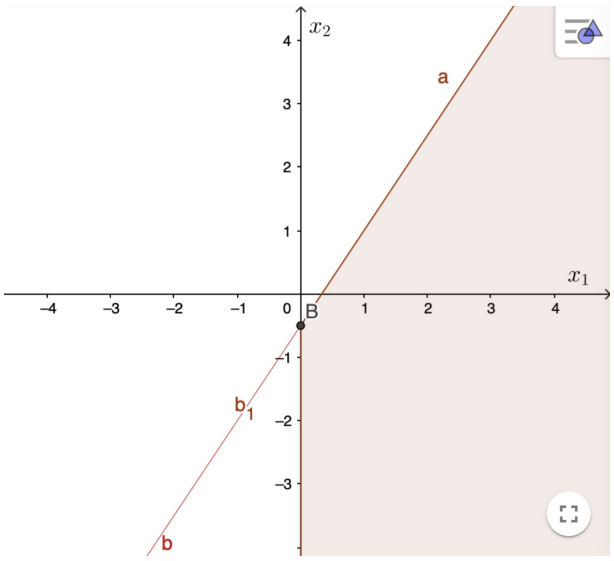
$$\mathbf{a} \leq \mathbf{b} \iff a_i \leq b_i \ \forall i = 1, 2, \ldots d. \tag{9}$$

Suppose we have a matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $\mathbf{b} \in \mathbb{R}^2$ as follows.

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix}, \mathbf{b} = [1, 0]. \tag{10}$$

Visualize the set of vectors where $A\mathbf{x} \geq \mathbf{b}$. Hint: A matrix vector product is a collection of dot products, and the above set can be obtained by the intersection of two of the sets constructed in the previous questions.

**Solution** $A\mathbf{x} = [3x_1 - 2x_2, 2x_1] \geq [1, 0]$, so it's the intersection of $3x_1 - 2x_2 \geq 1$ and $x_1 \geq 0$




## 4) More gradient computations

(i) Suppose we have the following loss function.

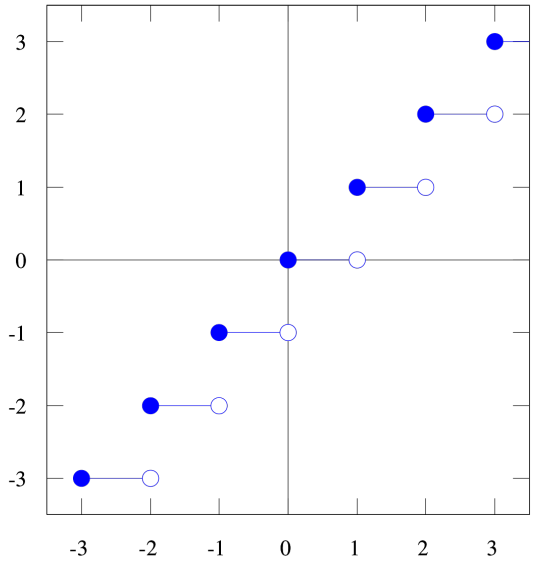
$$\text{Loss}(x, y, \mathbf{w}) = \max\{1 - \lfloor(\mathbf{w} \cdot \phi(x))y\rfloor, 0\}, \tag{11}$$

where $\lfloor a \rfloor$ returns $a$ rounded down to the nearest integer. Determine what the gradient of this function looks like, and whether gradient descent is suitable to optimize this loss function.

**Solution**

$$\text{Loss}(x, y, \mathbf{w}) = \begin{cases} 1 - \lfloor(\mathbf{w} \cdot \phi(x))y\rfloor & \text{if } \lfloor(\mathbf{w} \cdot \phi(x))y\rfloor \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

If we draw the plot for the floor function, we can see that its derivative is 0 (the lines are flat and the slope is 0) almost everywhere.




Thus, when applying chain rule to find the gradient of $\text{Loss}(x, y, \mathbf{w})$, the computed gradient will also be 0 almost everywhere, so gradient descent is not suitable to optimize this function as the iterates would not move from the point of initialization.

(ii) Compute the gradient of the loss function below.

$$\text{Loss}(x, y, \mathbf{w}) = \sigma(-(\mathbf{w} \cdot \phi(x))y), \tag{13}$$

where $\sigma(z) = (1 + \exp(-z))^{-1}$ is the logistic function.

**Solution** Let $z = (-\mathbf{w} \cdot \phi(x))y$, then $\text{Loss}(x, y, \mathbf{w}) = \sigma(z) = (1 + \exp(-z))^{-1}$. Applying the chain rule, we get

$$\begin{aligned}
\nabla_{\mathbf{w}} \text{Loss}(x, y, \mathbf{w}) &= \frac{\partial \sigma(z)}{\partial z} \nabla_{\mathbf{w}} z && (14)\\
&= -(1 + \exp(-z))^{-2} \exp(-z) y \phi(x) && (15)\\
&= -(1 + \exp(-z))^{-1} \Big( \frac{\exp -z}{1 + \exp(-z)} \Big) y \phi(x) && (16)\\
&= -\sigma(z)(1 - \sigma(z)) y \phi(x). && (17)
\end{aligned}$$

Plugging in the expression for $z$ gives us the final expression.

$$\nabla_{\mathbf{w}} \text{Loss}(x, y, \mathbf{w}) = -\sigma(-(\mathbf{w} \cdot \phi(x))y)(1 - \sigma(-(\mathbf{w} \cdot \phi(x))y)) y \phi(x). \tag{18}$$