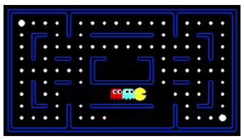
Games: non-zero-sum games



Utility functions

Competitive games: minimax (linear programming)

Collaborative games: pure maximization (plain search)

Real life: ?

- So far, we have focused on competitive games, where the utility of one player is the exact opposite of the utility of the other. The minimax principle is the appropriate tool for modeling these scenarios.
- On the other extreme, we have collaborative games, where the two players have the same utility function. This case is less interesting, because we are just doing pure maximization (e.g., finding the largest element in the payoff matrix or performing search).
- In many practical real life scenarios, games are somewhere in between pure competition and pure collaboration. This is where things get interesting...

Prisoner's dilemma

Example: Prisoner's dilemma

Prosecutor asks A and B individually if each will testify against the other.

If both testify, then both are sentenced to 5 years in jail.

If both refuse, then both are sentenced to 1 year in jail.

If only one testifies, then he/she gets out for free; the other gets a 10-year sentence.

[play with a partner]

answer in chat

Question

What was the outcome?

- player A testified, player B testified
- player A refused, player B testified
- player A testified, player B refused
- player A refused, player B refused

Prisoner's dilemma

Example: payoff matrix

B \ A	testify	refuse
testify	$A = -5, B = -5$	$A = -10, B = 0$
refuse	$A = 0, B = -10$	$A = -1, B = -1$

Definition: payoff matrix

Let $V_p(\pi_{\mathsf{A}}, \pi_{\mathsf{B}})$ be the utility for player p.

- In the prisoner's dilemma, the players get both penalized only a little bit if they both refuse to testify, but if one of them defects, then the other will get penalized a huge amount. So in practice, what tends to happen is that both will testify and both get sentenced to 5 years, which is clearly worse than if they both had cooperated.

Nash equilibrium

Can't apply von Neumann's minimax theorem (not zero-sum), but get something weaker:

Definition: Nash equilibrium

A **Nash equilibrium** is $(\pi^*_{\mathsf{A}}, \pi^*_{\mathsf{B}})$ such that no player has an incentive to change his/her strategy:

$V_{\mathsf{A}}(\pi^*_{\mathsf{A}}, \pi^*_B) \geq V_{\mathsf{A}}(\pi_{\mathsf{A}}, \pi^*_{\mathsf{B}})$ for all π_A

$V_{\mathsf{B}}(\pi^*_{\mathsf{A}}, \pi^*_B) \geq V_{\mathsf{B}}(\pi^*_{\mathsf{A}}, \pi_{\mathsf{B}})$ for all π_B

Theorem: Nash's existence theorem [1950]

In any finite-player game with finite number of actions, there exists **at least one** Nash equilibrium.

- Since we no longer have a zero-sum game, we cannot apply the minimax theorem, but we can still get a weaker result.
- A Nash equilibrium is kind of a state point, where no player has an incentive to change his/her policy unilaterally. Another major result in game theory is Nash's existence theorem, which states that any game with a finite number of players (importantly, not necessarily zero-sum) has at least one Nash equilibrium (a stable point). It turns out that finding one is hard, but we can be sure that one exists.

Examples of Nash equilibria

Example: Two-finger Morra

Nash equilibrium: A and B both play $\pi = [\frac{7}{12}, \frac{5}{12}]$.

Example: Collaborative two-finger Morra

Two Nash equilibria:

- A and B both play 1 (value is 2).
- A and B both play 2 (value is 4).

Example: Prisoner's dilemma

Nash equilibrium: A and B both testify.

- Here are three examples of Nash equilibria. The minimax strategies for zero-sum are also equilibria (and they are global optima).
- For purely collaborative games, the equilibria are simply the entries of the payoff matrix for which no other entry in the row or column are larger. There are often multiple local optima here.
- In the Prisoner's dilemma, the Nash equilibrium is when both players testify. This is of course not the highest possible reward, but it is stable in the sense that neither player would want to change his/her strategy. If both players had refused, then one of the players could testify to improve his/her payoff (from -1 to 0).

Summary so far

Simultaneous zero-sum games:

- von Neumann's minimax theorem
- Multiple minimax strategies, single game value

Simultaneous non-zero-sum games:

- Nash's existence theorem
- Multiple Nash equilibria, multiple game values

Huge literature in game theory / economics

- For simultaneous zero-sum games, all minimax strategies have the same game value (and thus it makes sense to talk about the value of a game). For non-zero-sum games, different Nash equilibria could have different game values (for example, consider the collaborative version of two-finger Morra).