# Logic: modus ponens with Horn clauses

# Definite clauses

**Definition: Definite clause**

A **definite clause** has the following form:

$$(p_1 \wedge \cdots \wedge p_k) \rightarrow q$$

where $p_1, \dots, p_k, q$ are propositional symbols.

Intuition: if $p_1, \dots, p_k$ hold, then $q$ holds.

Example: (Rain $\wedge$ Snow) $\rightarrow$ Traffic

Example: Traffic

Non-example: $\neg$Traffic

Non-example: (Rain $\wedge$ Snow) $\rightarrow$ (Traffic $\vee$ Peaceful)

- First we will choose to restrict the allowed set of formulas. Towards that end, let's define a **definite clause** as a formula that says, if a conjunction of propositional symbols holds, then some other propositional symbol $q$ holds. Note that this is a formula, not to be confused with an inference rule.

# Horn clauses

**Definition: Horn clause**

A **Horn clause** is either:

- a definite clause $(p_1 \wedge \cdots \wedge p_k \to q)$
- a goal clause $(p_1 \wedge \cdots \wedge p_k \to \text{false})$

Example (definite): $(\text{Rain} \wedge \text{Snow}) \to \text{Traffic}$

Example (goal): $\text{Traffic} \wedge \text{Accident} \to \text{false}$

equivalent: $\neg(\text{Traffic} \wedge \text{Accident})$

- A **Horn clause** is basically a definite clause, but includes another type of clause called a **goal clause**, which is the conjunction of a bunch of propositional symbols implying false. The form of the goal clause might seem a bit strange, but the way to interpret it is simply that it's the negation of the conjunction.

# Modus ponens

Inference rule:

**Definition: Modus ponens**

$$\frac{p_1, \quad \cdots \quad, p_k, \quad (p_1 \wedge \cdots \wedge p_k) \to q}{q}$$

Example:

**Example: Modus ponens**

$$\frac{\text{Wet}, \quad \text{Weekday}, \quad \text{Wet} \wedge \text{Weekday} \to \text{Traffic}}{\text{Traffic}}$$

- Recall the Modus ponens rule from before. We simply have generalized it to arbitrary number of premises.

# Completeness of modus ponens

**Theorem: Modus ponens on Horn clauses**

Modus ponens is **complete** with respect to Horn clauses:

- Suppose KB contains only Horn clauses and $p$ is an entailed propositional symbol.
- Then applying modus ponens will derive $p$.

Upshot:

KB $\models p$ (entailment) is the same as KB $\vdash p$ (derivation)!

- There's a theorem that says that modus ponens is complete on Horn clauses. This means that any propositional symbol that is entailed can be derived by modus ponens too, provided that all the formulas in the KB are Horn clauses.
- We already proved that modus ponens is sound, and now we have that it is complete (for Horn clauses). The upshot of this is that entailment (a semantic notion, what we care about) and being able to derive a formula (a syntactic notion, what we do with inference) are equivalent!

# Example: Modus ponens

**KB**

Rain

Weekday

Rain $\rightarrow$ Wet

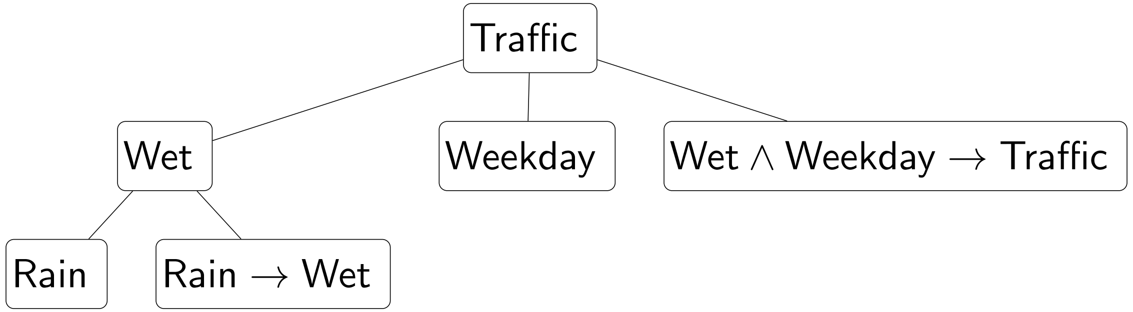
Wet $\wedge$ Weekday $\rightarrow$ Traffic

Traffic $\wedge$ Careless $\rightarrow$ Accident

**Definition: Modus ponens**

$$\frac{p_1, \quad \cdots \quad, p_k, \quad (p_1 \wedge \cdots \wedge p_k) \rightarrow q}{q}$$

Question: KB $\models$ Traffic $\quad\Leftrightarrow\quad$ KB $\vdash$ Traffic



- Let's see modus ponens on Horn clauses in action. Suppose we have the given KB consisting of only Horn clauses (in fact, these are all definite clauses), and we wish to ask whether the KB entails Traffic.
- We can construct a **derivation**, a tree where the root formula (e.g., Traffic) was derived using inference rules.
- The leaves are the original formulas in the KB, and each internal node corresponds to a formula which is produced by applying an inference rule (e.g., modus ponens) with the children as premises.
- If a symbol is used as the premise in two different rules, then it would have two parents, resulting in a DAG.

# Summary