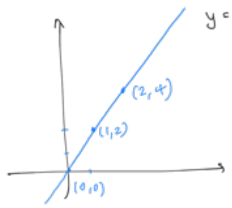
Basics: differentiation

① Real function over reals

$f: \mathbb{R} \to \mathbb{R} \qquad y = f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



$\frac{\Delta y}{\Delta x}$ at $(0,0)$

(i) $\Delta x = 1 \qquad \Delta y = 2$

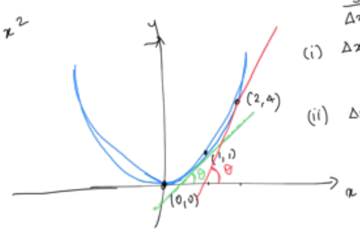
$\frac{\Delta y}{\Delta x} = 2$

(ii) $\Delta x = 2 \qquad \Delta y = 4$

$\frac{\Delta y}{\Delta x} = 2$

$$\boxed{\frac{\Delta y}{\Delta x} = 2}$$

$y = x^2$



$\frac{\Delta y}{\Delta x}$ at $(1,1)$

(i) $\Delta x = 1 \qquad \frac{\Delta y}{\Delta x} = 3$

(ii) $\Delta x = -1 \qquad \frac{\Delta y}{\Delta x} = 1$

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Examples

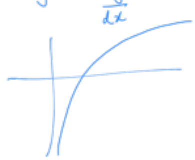
① $y = c \qquad \frac{dy}{dx} = 0$

② $y = \alpha x \qquad \frac{dy}{dx} = \alpha$

③ $y = x^n \qquad \frac{dy}{dx} = n x^{n-1}$

$y = x^2 \quad \frac{dy}{dx} = 2x$

④ $y = \log x \qquad \frac{dy}{dx} = \frac{1}{x}$



⑤ $y = e^x \qquad \frac{dy}{dx} = e^x$

Sum rules:

$y = f(x) + g(x)$

$$\frac{dy}{dx} = \frac{d f(x)}{dx} + \frac{d g(x)}{dx}$$

$$y = \sum_{i=1}^{n} f_i(x) \qquad \frac{dy}{dx} = \sum_{i=1}^{n} \frac{d f_i(x)}{dx}$$

Chain Rule:

$y = f(u) \qquad u = g(x)$

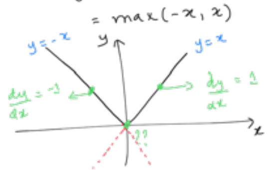
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex: $y = e^{4x}$ $\quad u = 4x$

$$\frac{dy}{x} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{4x} \cdot 4$$

Q: Does derivative always exist?

$$y = \max\left(f(x), g(x)\right)$$
$$= \max(-x, x) \qquad |x|$$



Partial derivative

$$y = f(x, z) \qquad x, z \in \mathbb{R}$$

change in y only due to change in z

$$\frac{\partial y}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, z+\Delta z) - f(x,z)}{\Delta z}$$

$$\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, z) - f(x,z)}{\Delta x}$$

Ex: $y = 4xz^2$

$$\frac{\partial y}{\partial x} = \frac{\partial (4z^2)x}{\partial x} = 4z^2$$

$$\frac{\partial y}{\partial z} = \frac{\partial}{\partial z}(4x)z^2 = (4x)\cdot 2z = 8xz$$

Real functions of vectors

$$v \in \mathbb{R}^d \qquad v = (v_1, v_2 \dots v_d)$$
$$y = f(v) \qquad y \in \mathbb{R}$$

Notation: "gradient": $\nabla_v f(v)$

$$\nabla_v f(v) \in \mathbb{R}^d = \left(\frac{\partial f(v)}{\partial v_1}, \frac{\partial f(v)}{\partial v_2} \dots \frac{\partial f(v)}{\partial v_d} \right)$$

Ex: ① $f(v) = c \cdot v \qquad \nabla_v f(v)$

$$c \cdot v = \sum_{i=1}^{d} c_i v_i$$

$$\frac{\partial f(v)}{\partial v_i} = c_i$$

$$\nabla_v f(v) = (c_1, c_2 \dots c_d)$$
$$= c$$

② $f(v) = \|v\|^2 = \sum_{i=1}^{d} v_i^2$

$$\nabla_v f(v) = ? \qquad \frac{\partial f(v)}{\partial v_i} = 2v_i$$

$$\nabla_v f(v) = (2v_1, 2v_2 \dots 2v_d)$$
$$= 2v$$

③ $f(v) = v^T A v \qquad \nabla_v f(v) = (A + A^T)$

Real functions of matrices

$$P \in \mathbb{R}^{m \times n} \qquad P = \begin{pmatrix} - P_1 - \\ - P_2 - \\ \vdots \\ - P_m - \end{pmatrix}$$

$f(P) \in \mathbb{R}$

$$\nabla_P f(P) = \begin{pmatrix} — & \nabla_{P_1} f(P) & — \\ — & \nabla_{P_2} f(P) & — \\ & \vdots & \end{pmatrix}$$

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & \\ P_{21} & P_{22} & \cdots & \\ & & & P_{mn} \end{pmatrix}$$

$P_{ij} =$ i^{th} vector $[P_i]$
j^{th} component of P^i

$$\nabla_P f(P) = \begin{pmatrix} \frac{\partial f(P)}{\partial P_{11}} & \frac{\partial f(P)}{\partial P_{12}} & \cdots \\ \vdots & & \frac{\partial f(P)}{\partial P_{mn}} \end{pmatrix}$$