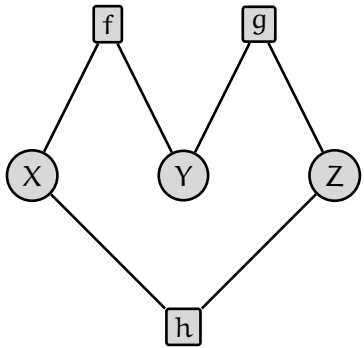
Variable Elimination and Treewidth

CS 221: Finn & Anari

Factor Graph Review



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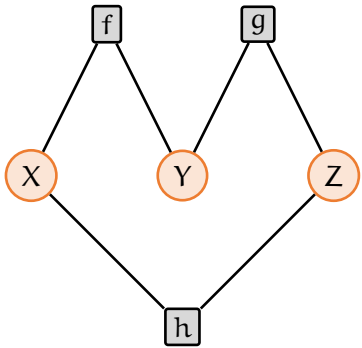

Variables

- The X, Y, Z nodes.
- Each variable takes values from a specified set called its domain.

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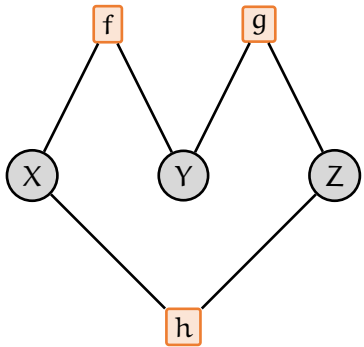


Variables

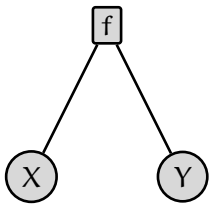
- The X, Y, Z nodes.
- Each variable takes values from a specified set called its domain.

Factors

- The f, g, h nodes.
- Each factor is a local function whose input consists of values for connected variables.

Weight Database

- Each factor is a small table indexed by connected variables.



X	Y	f
0	0	1.5
0	1	2.5
1	0	0.3
1	1	1.9

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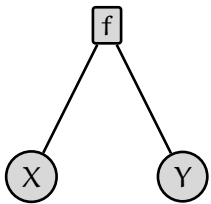

X	Y	f
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- Together they form the global weight database.

X	Y	Z	weight $= f \cdot g \cdot h$
0	0	0	$f(0,0) \cdot g(0,0) \cdot h(0,0)$
0	0	1	$f(0,0) \cdot g(0,1) \cdot h(0,1)$
0	1	0	$f(0,1) \cdot g(1,0) \cdot h(0,0)$
0	1	1	$f(0,1) \cdot g(1,1) \cdot h(0,1)$
1	0	0	$f(1,0) \cdot g(0,0) \cdot h(1,0)$
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- Weight database is exponentially large. 

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- Factors are more manageable as long as their arity is small.

$$\begin{aligned} \text{unary factor} &\implies 2 \text{ rows} \\ \text{binary factor} &\implies 4 \text{ rows} \\ &\vdots \end{aligned}$$

Inference Problem

- **Input:** Description of factors.
- **Output:** Weight database?

Inference Problem

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Inference Problem

- **Input:** Description of factors. 
- **Output:** Reduction of weight database. 

Reduction

Key Idea

Produce table on just a small subset of variables.

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Using a reduction/aggregation operator.

Sum ($\sum$)

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- Partial reduction (subset $\neq \emptyset$):

$$\text{output}(x) := \sum_{y} \sum_{z} \text{weight}(x, y, z)$$

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- Partial reduction (subset $\neq \emptyset$):

$$\text{output}(x) := \sum_{y} \sum_{z} \text{weight}(x, y, z)$$

Max ($\bigvee$)

- Full reduction (subset $= \emptyset$):

$$\text{output} = \bigvee_{x} \bigvee_{y} \bigvee_{z} \text{weight}(x, y, z)$$

global weight database

X	Y	Z	weight $= f \cdot g \cdot h$
0	0	0	$f(0,0) \cdot g(0,0) \cdot h(0,0)$
0	0	1	$f(0,0) \cdot g(0,1) \cdot h(0,1)$
0	1	0	$f(0,1) \cdot g(1,0) \cdot h(0,0)$
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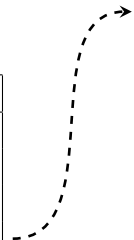

X	Y	output
0	0	$\sum_z f(0,0) \cdot g(0,z) \cdot h(0,z)$
0	1	$\sum_z f(0,1) \cdot g(1,z) \cdot h(0,z)$
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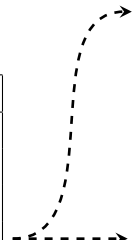

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X	output
0	$\sum_y \sum_z f(0,y) \cdot g(y,z) \cdot h(0,z)$
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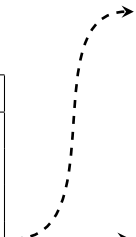

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output
$\sum_x \sum_y \sum_z f(x,y) \cdot g(y,z) \cdot h(x,z)$

Reducing n variables with $\{0, 1\}$ domains naively takes $\geqslant 2^n$ time. 

$$\underbrace{\sum_x \sum_y \sum_z \cdots}_{2^n \text{ terms in these sums}}$$

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Can we do better?

Variable Elimination

$$\sum_{x} \sum_{y} \sum_{z} f(x, y) \cdot g(y, z) \cdot h(x, z)$$

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- Precompute its table by enumerating y, z, and each time summing over x.

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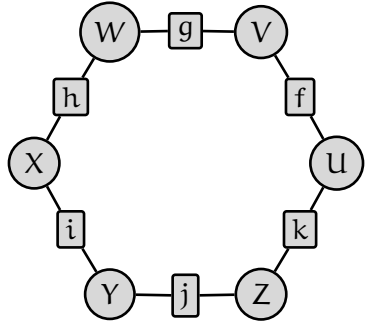
- For this particular example, this does not seem like much help. 

Example: Cycle Factor Graph



$$\sum_{u}\sum_{v}\sum_{w}\sum_{x}\sum_{y}\sum_{z} f(u,v)\cdot g(v,w)\cdot h(w,x)\cdot i(x,y)\cdot j(y,z)\cdot k(z,u)$$

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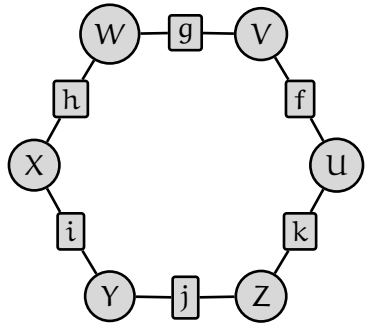

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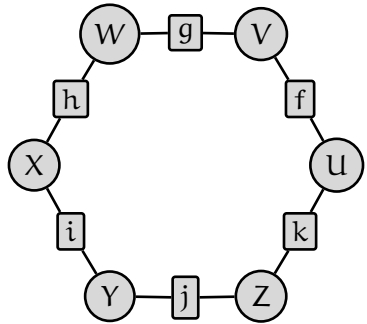

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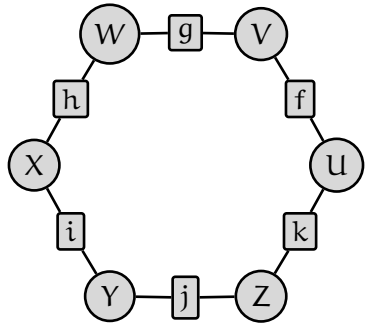

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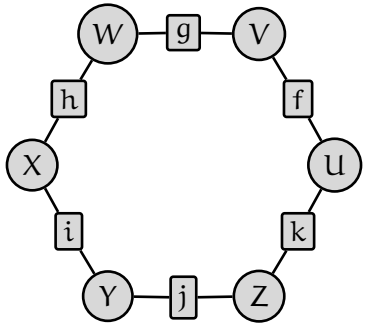

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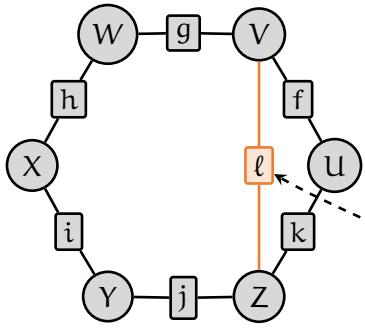

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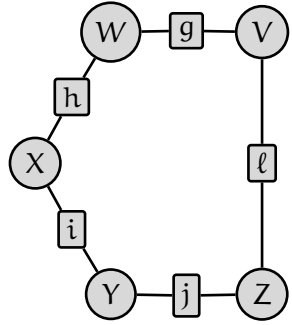

$$\sum_{v}\sum_{w}\sum_{x}\sum_{y}\sum_{z} g(v,w)\cdot h(w,x)\cdot i(x,y)\cdot j(y,z)\cdot \ell(v,z)$$

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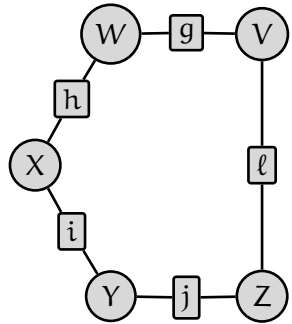

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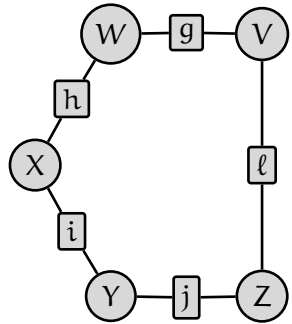

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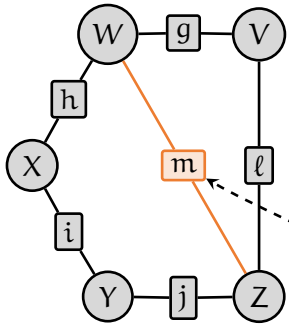

$$\sum_{w}\sum_{x}\sum_{y}\sum_{z} h(w,x)\cdot i(x,y)\cdot j(y,z)\cdot \overbrace{\left(\sum_{v} g(v,w)\cdot \ell(v,z)\right)}$$

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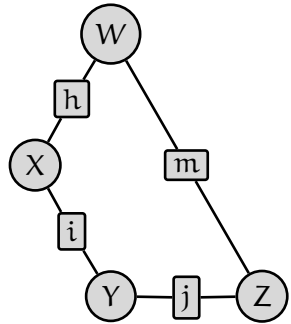

$$\sum_{w}\sum_{x}\sum_{y}\sum_{z} \mathfrak{h}(w,x)\cdot \mathfrak{i}(x,y)\cdot \mathfrak{j}(y,z)\cdot \mathfrak{m}(w,z)$$

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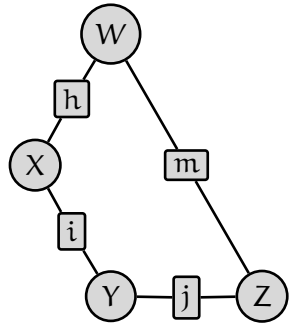

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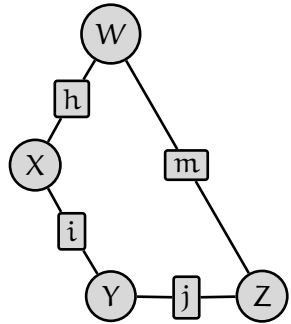

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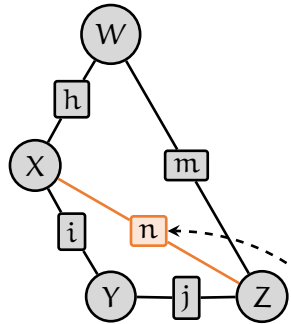

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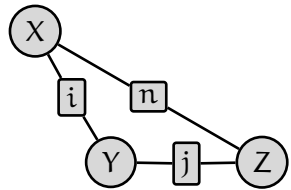

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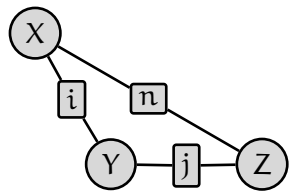

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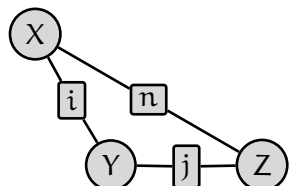
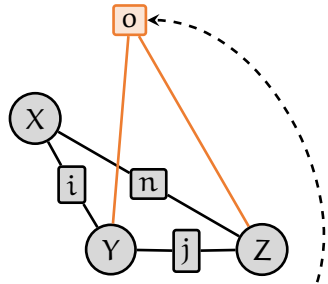


$$\sum_{y} \sum_{z} j(y,z) \cdot \left(\sum_{x} i(x,y) \cdot n(x,z) \right)$$

Example: Cycle Factor Graph



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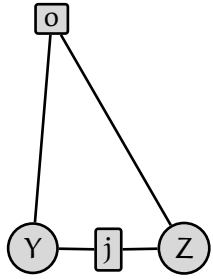

$$\sum_{y} \sum_{z} j(y,z) \cdot o(y,z)$$

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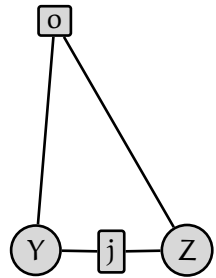

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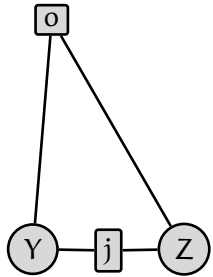

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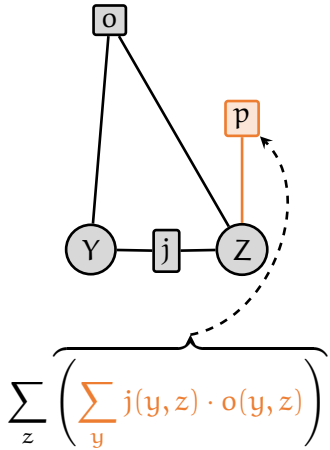

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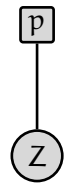

$$\sum_{z} \mathrm{p}(z)$$

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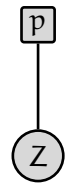

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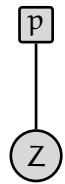

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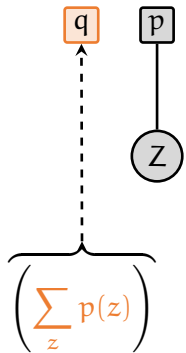

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q

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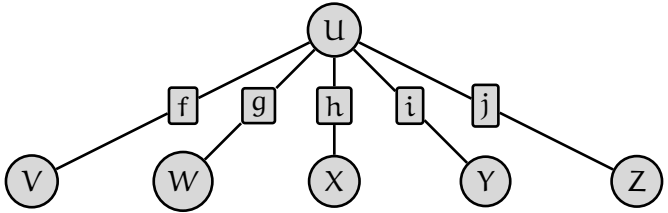
- In general, performance depends on the order of eliminating variables (not for the cycle graph though).


U
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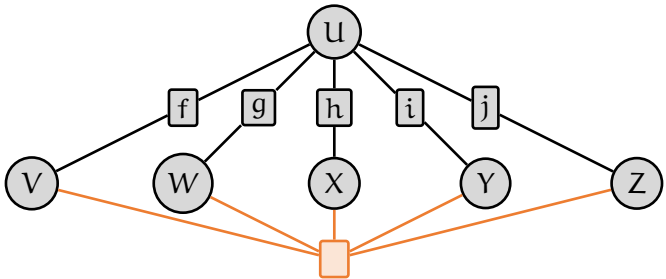
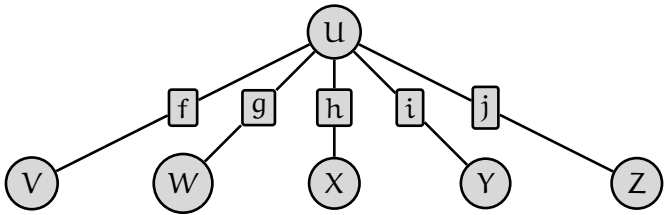
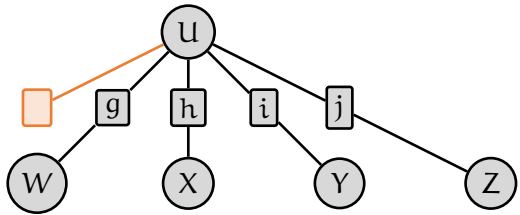


- If we eliminate u first, we create a 5-ary factor. ☹



- If we eliminate U first, we create a 5-ary factor. ☹
- If we eliminate V, W, X, Y, Z and then U, we never create more than unary factors. ☺



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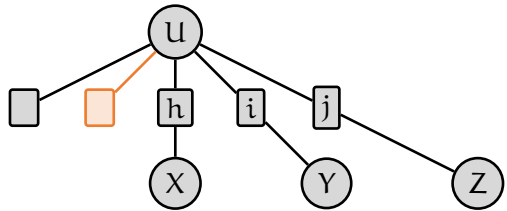

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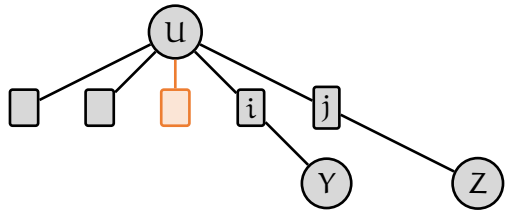

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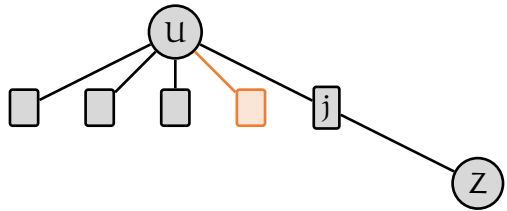
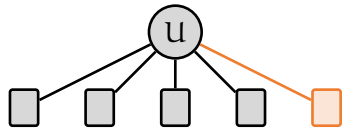


- If we eliminate U first, we create a 5-ary factor. 😕
- If we eliminate V, W, X, Y, Z and then U, we never create more than unary factors. 🙂



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Treewidth

The best case measure of performance, among all orderings.

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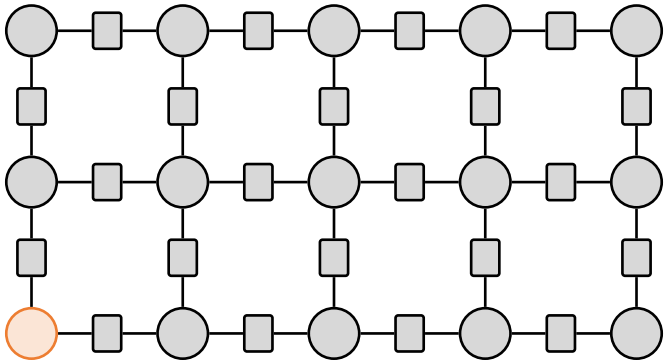
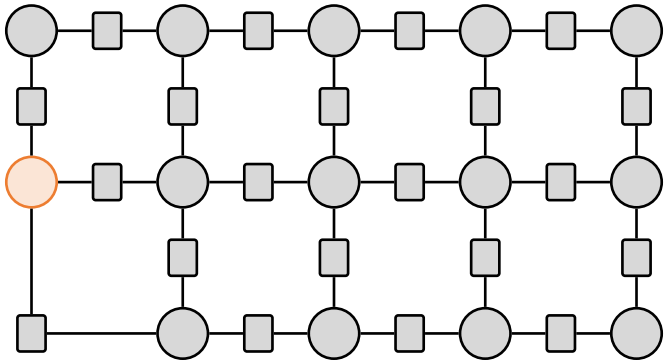
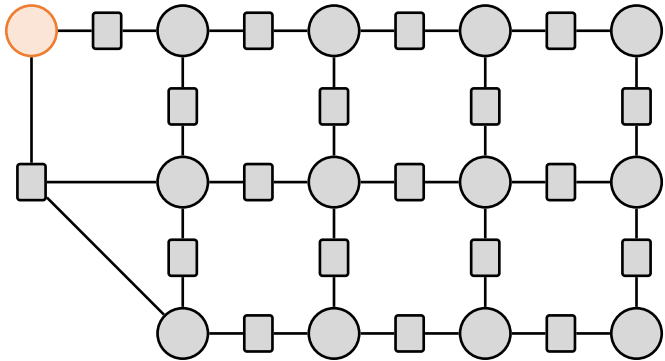
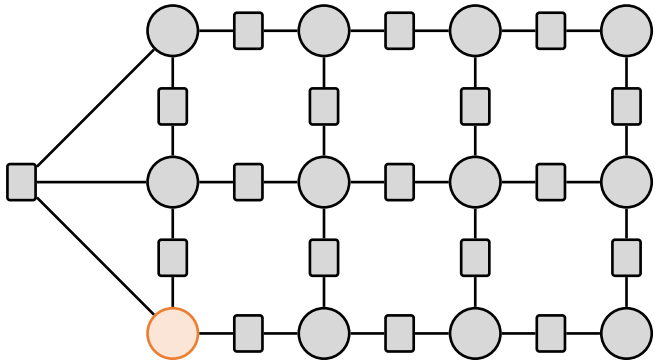
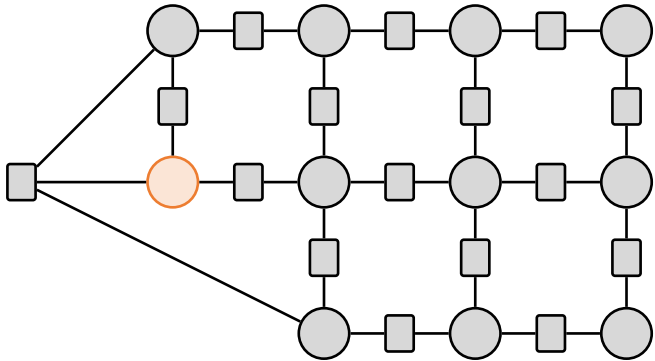
- It is NP-hard to compute treewidth. 
- Greedy heuristic: Eliminate the variable with the least number of neighboring variables each time.
- If the factor graph has no loop (i.e., a tree), the greedy algorithm is provably optimal. It only creates unary factors. 

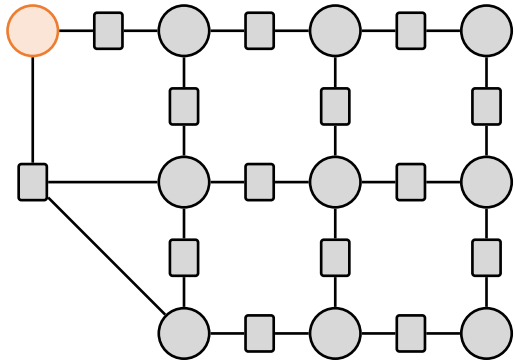


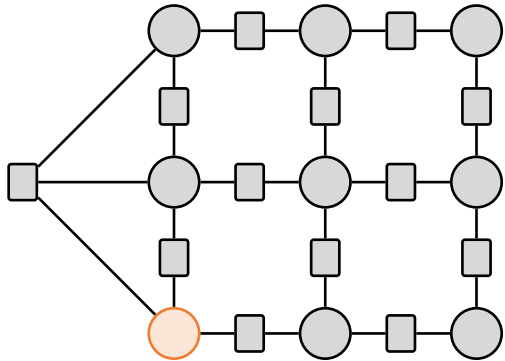


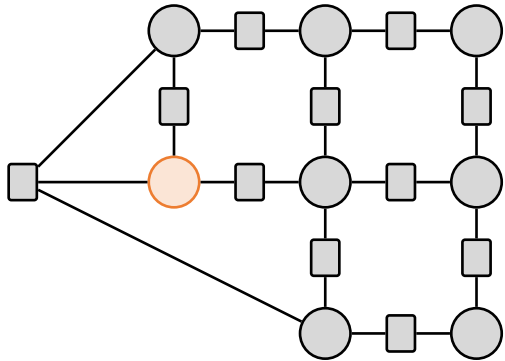


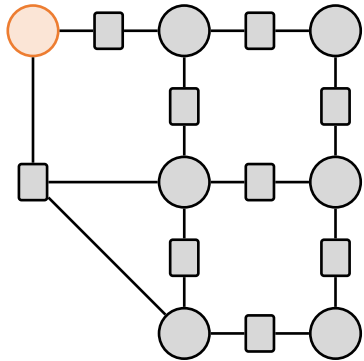


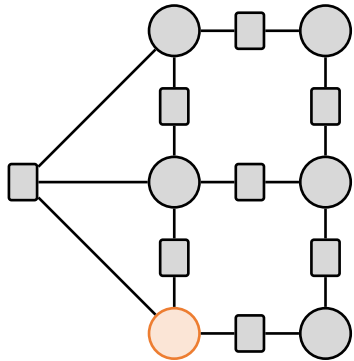


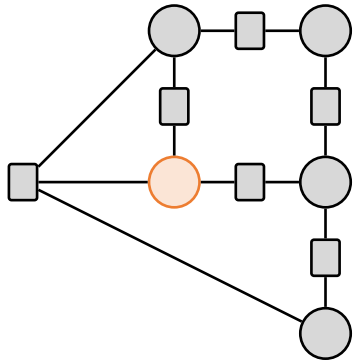


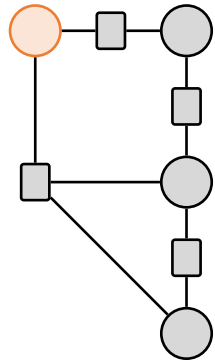


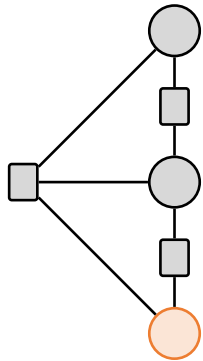


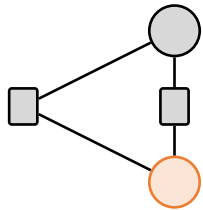


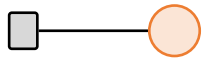












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