CS221 Problem Workout

Week 2

1) [CA session] Problem 1: Least-Squares Linear Regression

In last week's module we studied the linear regression algorithm, which solves a regression problem using a linear predictor via optimizing the objective

$$\text{TrainLoss}(\mathbf{w}) = \frac{1}{|\mathcal{D}_{\text{train}}|} \sum_{(\mathbf{x},y)\in\mathcal{D}_{\text{train}}} (\mathbf{w}\cdot\phi(\mathbf{x}) - y)^2. \tag{1}$$

The training loss was minimized via gradient descent, which works iteratively to decrease the training loss. As mentioned in the module, we can actually solve for the optimal weights $\mathbf{w}^\star$ in closed-form. In this problem we will derive the *normal equations* used to solve for this estimator.

2) [CA session] Problem 2: Non-linear features

Consider the following two training datasets of (x, y) pairs:

- $\mathcal{D}_1 = \{(-1, +1), (0, -1), (1, +1)\}$.
- $\mathcal{D}_2 = \{(-1, -1), (0, +1), (1, -1)\}$.

Observe that neither dataset is linearly separable if we use $\phi(x) = x$, so let's fix that.

Define a two-dimensional feature function $\phi(x)$ such that:

- There exists a weight vector $\mathbf{w}_1$ that classifies $\mathcal{D}_1$ perfectly (meaning that $\mathbf{w}_1 \cdot \phi(x) > 0$ if x is labeled $+1$ and $\mathbf{w}_1 \cdot \phi(x) < 0$ if x is labeled -1); and
- There exists a weight vector $\mathbf{w}_2$ that classifies $\mathcal{D}_2$ perfectly.

Note that the weight vectors can be different for the two datasets, but the features $\phi(x)$ must be the same.

Some additional food for thought: Is every dataset linearly separable in some feature space? In other words, given pairs $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$, can we find a feature extractor ϕ such that we can perfectly classify $(\phi(\mathbf{x}_1), y_1), \ldots, (\phi(x_n), y_n)$ for some linear model $\mathbf{w}$? If so, is this a good feature extractor to use?

3) [CA session] Problem 3: Backpropagation

Consider the following function

$$\text{Loss}(x, y, z, w) = 2(xy + \max\{w, z\})$$

Run the backpropagation algorithm to compute the four gradients (each with respect to one of the individual variables) at $x = 3,\ y = -4,\ z = 2$ and $w = -1$. Use the following nodes: addition, multiplication, max, multiplication by a constant.

4) [breakout, optional] Problem 4: Non-linear decision boundaries

Suppose we are performing classification where the input points are of the form $(x_1, x_2) \in \mathbb{R}^2$. We can choose any subset of the following set of features:

$$\mathcal{F} = \left\{ x_1^2, x_2^2, x_1 x_2, x_1, x_2, \frac{1}{x_1}, \frac{1}{x_2}, 1, \mathbf{1}[x_1 \geq 0], \mathbf{1}[x_2 \geq 0] \right\} \tag{2}$$

For each subset of features $F \subseteq \mathcal{F}$, let $D(F)$ be the set of all decision boundaries corresponding to linear classifiers that use features F.

For each of the following sets of decision boundaries E, provide the minimal F such that $D(F) \supseteq E$. If no such F exists, write 'none'.

- E is all lines [CA hint]:

  ______________________________ (3)

- E is all circles centered at the origin:

  ______________________________ (4)

- E is all circles:

  ______________________________ (5)

- E is all axis-aligned rectangles:

  ______________________________ (6)

- E is all axis-aligned rectangles whose lower-right corner is at $(0, 0)$:

  ______________________________ (7)

5) [breakout, optional] Problem 5: K-means

Consider doing ordinary K-means clustering with $K = 2$ clusters on the following set of 3 one-dimensional points:

$$\{-2, 0, 10\}. \tag{8}$$

Recall that K-means can get stuck in local optima. Describe the precise conditions on the initialization $\mu_1 \in \mathbb{R}$ and $\mu_2 \in \mathbb{R}$ such that running K-means will yield the global optimum of the objective function. Notes:

- Assume that $\mu_1 < \mu_2$.
- Assume that if in step 1 of K-means, no points are assigned to some cluster j, then in step 2, that centroid μ_j is set to ∞.
- Hint: try running K-means from various initializations μ_1, μ_2 to get some intuition; for example, if we initialize $\mu_1 = 1$ and $\mu_2 = 9$, then we converge to $\mu_1 = -1$ and $\mu_2 = 10$.