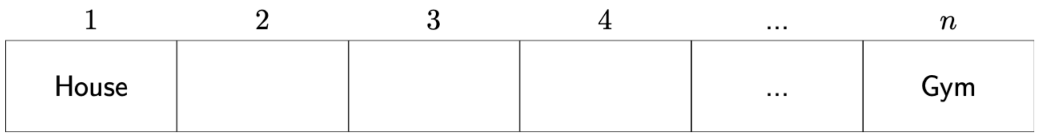
CS221 Problem Workout Solutions

1) [CA session] Problem 1

(a) Sabina wants to go from her house (located at 1) to the gym (located at n). At each location s, she can either (i) deterministically walk forward to the next location $s+1$ (takes 1 unit of time) or (ii) wait for the bus. The bus comes with probability ϵ, in which case, she will reach the gym in $1+\alpha(n-s)$ units of time, where α is some parameter. If the bus doesn't come, well, she stays put, and that takes 1 unit of time.



We have formalized the problem as an MDP for you:

- State: $s \in \{1, 2, \dots, n\}$ is Sabina's location
- $\text{Actions}(s) = \{\text{Walk}, \text{Bus}\}$
- $\text{Reward}(s, \text{Walk}, s') = \begin{cases} -1 & \text{if } s' = s+1 \\ -\infty & \text{otherwise} \end{cases}$
- $\text{Reward}(s, \text{Bus}, s') = \begin{cases} -1-\alpha(n-s) & \text{if } s' = n \\ -1 & \text{if } s' = s \\ -\infty & \text{otherwise} \end{cases}$
- $T(s'|s, \text{Walk}) = \begin{cases} 1 & \text{if } s' = s+1 \\ 0 & \text{otherwise} \end{cases}$
- $T(s'|s, \text{Bus}) = \begin{cases} \epsilon & \text{if } s' = n \\ 1-\epsilon & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases}$
- $\text{IsEnd}(s) = \mathbf{1}[s = n]$

Compute a closed form expression for the value of the "always walk" policy and the "always wait for the bus" policy (using some or all of the variables ϵ, α, n). Assume a discount rate of $\gamma = 1$.

- $V_{\text{Walk}}(s) =$ ____

- $V_{\text{Bus}}(s) =$ ____

- For what values of ϵ (as a function of α and n) is it advantageous to walk rather than take the bus?

Solution Expected value for always walking:

$$V_{\text{Walk}} = -(n-s).$$

Expected value for always waiting for bus:

$$V_{\text{Bus}}(s) = \epsilon(-1-\alpha(n-s)) + (1-\epsilon)(-1+V_{\text{Bus}}(s)).$$

Simplifying, we get:

$$V_{\text{Bus}}(s) = -\alpha(n-s) - \frac{1}{\epsilon}.$$

For walking to be preferable, we need $V_{\text{Walk}}(s) \geq V_{\text{Bus}}(s)$, or equivalently:

$$n-s \leq \alpha(n-s) + \frac{1}{\epsilon} \Leftrightarrow (1-\alpha)(n-s) \leq \frac{1}{\epsilon} \Leftrightarrow \begin{cases} \epsilon \leq \frac{1}{(1-\alpha)(n-s)} & , \alpha < 1 \\ \epsilon > 0 & , \alpha \geq 1. \end{cases}$$

(b) Not surprisingly, buses operate strangely in this town, and we will now assume instead that Sabina doesn't know the reward function nor the transition probabilities. She decides to use reinforcement learning to find out. She starts by going around town using the two different modes of transportation:

s_0	a_1	r_1	s_1	a_2	r_2	s_2	a_3	r_3	s_3	a_4	r_4	s_4	a_5	r_5	s_5
1	Bus	-1	1	Bus	-1	1	Bus	3	3	Walk	1	4	Walk	1	5

Run the Q-learning algorithm once over the given data to compute an estimate of the optimal Q-value $Q_{\text{opt}}(s, a)$. Process the episodes from left to right. Assume all Q-values are initialized to zero, and use a learning rate of $\eta = 0.5$ and a discount of $\gamma = 1$.

- $\hat{Q}(1, \text{Walk}) =$ ________________________________

- $\hat{Q}(1, \text{Bus}) =$ ________________________________

- $\hat{Q}(3, \text{Walk}) =$ ________________________________

- $\hat{Q}(3, \text{Bus}) =$ ________________________________

- $\hat{Q}(4, \text{Walk}) =$ ________________________________

- $\hat{Q}(4, \text{Bus}) =$ ________________________________

Solution On each (s, a, r, s'), recall the Q-learning updates:

$$\hat{Q}_{opt}(s,a) \leftarrow (1-\eta)\hat{Q}_{opt}(s,a) + \eta(r + \gamma \max_{a' \in \text{Actions}(s')} \hat{Q}_{opt}(s', a')). \qquad (1)$$

Now the concrete updates:

- On $(1, \text{Bus}, -1, 1)$: $\hat{Q}(1, \text{Bus}) = 0.5(0) + 0.5(-1 + 1(\max(0, 0))) = -0.5$
- On $(1, \text{Bus}, -1, 1)$: $\hat{Q}(1, \text{Bus}) = 0.5(-0.5) + 0.5(-1 + 1(\max(0, -0.5))) = -0.75$
- On $(1, \text{Bus}, 3, 3)$: $\hat{Q}(1, \text{Bus}) = 0.5(-0.75) + 0.5(3 + 1(\max(0, 0))) = 1.125$
- On $(3, \text{Walk}, 1, 4)$: $\hat{Q}(3, \text{Walk}) = 0.5(0) + 0.5(1 + 1(\max(0, 0))) = 0.5$
- On $(4, \text{Walk}, 1, 5)$: $\hat{Q}(4, \text{Walk}) = 0.5(0) + 0.5(1 + 1(\max(0, 0))) = 0.5$

The final values:

- $\hat{Q}(1, \text{Walk}) = 0$
- $\hat{Q}(1, \text{Bus}) = 1.125$
- $\hat{Q}(3, \text{Walk}) = 0.5$
- $\hat{Q}(3, \text{Bus}) = 0$
- $\hat{Q}(4, \text{Walk}) = 0.5$
- $\hat{Q}(4, \text{Bus}) = 0$

2) [Extra Practice] Problem 2

You're programming a self-driving car that can take you from home (position 1) to school (position n). At each time step, the car has a current position $x \in \{1, \ldots, n\}$ and a current velocity $v \in \{0, \ldots, m\}$. The car starts with $v = 0$, and at each time step, the car can either increase the velocity by 1, decrease it by 1, or keep it the same; this new velocity is used to advance x to the new position. The velocity is not allowed to exceed the speed limit m nor return to 0.

In addition, to prevent people from recklessly cruising down Serra Mall, the university has installed speed bumps at a subset of the n locations. The speed bumps are located at $B \subseteq \{1, \ldots, n\}$. The car is not allowed to enter, leave, or pass over a speed bump with velocity more than $k \in \{1, \ldots, m\}$. **Your goal is to arrive at position n with velocity 1 in the smallest number of time steps.**

Figure 1 shows an example with $n = 9$ positions and one speed bump $B = \{5\}$. If the maximum speed is $m = 3$ and $k = 1$ for a speed bump, then an example of a legal path is the following:

$$(1,0) \xrightarrow{+1} (2,1) \xrightarrow{+1} (4,2) \xrightarrow{-1} (5,1) \xrightarrow{0} (6,1) \xrightarrow{+1} (8,2) \xrightarrow{-1} (9,1)$$

$x = 1$ home	$x = 2$	$x = 3$	$x = 4$	$x = 5$ bump	$x = 6$	$x = 7$	$x = 8$	$x = 9$ school

Figure 1: An example of a legal path that takes 6 time steps with $m = 3$ and $k = 1$. We show the position-velocity pairs (x, v) at each time step, and each number above an arrow is an acceleration (change in velocity).

(a) It turns out that you were so excited about the AI algorithms that you didn't really pay much attention to the brakes of the car. As a result, when you try to decrease the velocity by 1, with some failure probability α, the velocity actually stays the same. To simplify our lives, assume there are no speed bumps. Assume a reward of R if we get to school (at a velocity of 1) but 0 if we pass the school, with a cost of 1 per time step. Let us formulate the resulting problem as an MDP:

- $s_{\text{start}} = (1, 0)$
- $\text{Actions}((x, v)) = \{a \in \{+1, 0, -1\} : x + v + a \leq n \wedge v + a \leq m \wedge (v + a \leq k \vee \{x, \ldots, x + v + a\} \cap B = \emptyset)\}$. Suppose we want to apply acceleration a. First, we want to make sure we don't exceed the school ($x + v + a \leq n$) or go out of the velocity range ($v + a \leq m$). Next, we want to make sure that we're not entering, passing through, or leaving any speed bumps at a velocity greater than k. This is captured logically by ensuring a safe speed ($v + a \leq k$) or checking that there are no speed bumps between x and the new location $x + v + a$.

- $T((x', v')|(x, v), a) =$ (to be filled out by you below)
- $\text{Reward}((x, v), a, (x', v')) = R \cdot \mathbb{1}[x' = n \wedge v' = 1] - 1$
- $\text{IsEnd}((x, v)) = \mathbb{1}[x \geq n]$

(i) Fill out the definition of the transition probabilities T:

$T((x', v')|(x, v), a) =$

Solution

$$T((x', v')|(x, v), a) = \begin{cases} \alpha & \text{if } x' = x + v' \text{ and } v' = v \text{ and } a = -1 \\ 1 - \alpha & \text{if } x' = x + v' \text{ and } v' = v + a \text{ and } a = -1 \\ 1 & \text{if } x' = x + v' \text{ and } v' = v + a \text{ and } a \neq -1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let us explore the effect of unreliable brakes. Consider the example in Figure2.

$x = 1$ home	$x = 2$	$x = 3$	$x = 4$	$x = 5$ school

Figure 2: An small driving environment without speed bumps.

Consider two policies:

- π_1: always move with velocity 1:

$$\pi_1((1,0)) = +1 \quad \pi_1((2,1)) = 0 \quad \pi_1((3,1)) = 0 \quad \pi_1((4,1)) = 0.$$

- π_2: speed up and slow down:

$$\pi_2((1,0)) = +1 \quad \pi_2((2,1)) = +1 \quad \pi_2((4,2)) = -1.$$

Compute the expected utility of π_1 as a function of α and R (with discount $\gamma = 1$).

Solution The policy π_1 deterministically obtains reward $\boxed{R-4}$. Using $Reward(x', v')$ we have $Reward(2,1) + Reward(3,1) + Reward(4,1) + Reward(5,1) = -1 - 1 - 1 - 1 + R$

Compute the expected utility of π_2 as a function of α and R (with discount $\gamma = 1$).

Solution The policy π_2 obtains reward $\boxed{(1-\alpha)R - 3}$. We only get reward R if we are able to break at the end so: $(1-\alpha)(Reward(2,1) + Reward(4,2) + Reward(5,1)) + \alpha$
$(Reward(2,1) + Reward(4,2) + Reward(6,2)) = (1-\alpha)(R-3) + \alpha(-3) = (1-\alpha)R - 3$

For what values of α and R does π_2 obtain higher expected reward than π_1? Your answer should be an expression relating α and R.

Solution Therefore, π_2 is better when $(1-\alpha)R - 3 > R - 4$, which is precisely when $\boxed{\alpha < 1/R}$.

(b) Bad news: you realize that your brakes are not only faulty, but that you don't know how often they fail (α is unknown). Circle all of the following algorithms that can be used to compute the optimal policy in this setting:

model-based value iteration   model-free Monte Carlo   SARSA   Q-learning

Solution **Model-based value iteration** would estimate the transition probabilities, which can be used to compute the optimal policy. **Q-learning** can be used to directly estimate the value of the optimal policy. Model-free Monte Carlo and SARSA can only be used to compute the value of a fixed policy.